Modular forms, modular symbols

(PARI-GP version 2.15.3)

Modular Forms

Dirichlet characters

Characters are encoded in three different ways:

• a `t_INT` $D \equiv 0, 1 \bmod 4$: the quadratic character $(D/\cdot)$;

• a `t_INTMOD` `Mod`(m, q), $m \in (\mathbf{Z}/q)^*$ using a canonical bijection with the dual group (the Conrey character $\chi_q(m, \cdot)$);

• a pair $[G, chi]$, where $G = znstar(q, 1)$ encodes $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \le k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$ and the vector $chi = [c_1, \ldots, c_k]$ encodes the character such that $\chi(g_j) = e(c_j/d_j)$.

initialize $G = (\mathbf{Z}/q\mathbf{Z})^*$	`G = znstar`$(q, 1)$
convert datum D to $[G, \chi]$	`znchar`(D)
Galois orbits of Dirichlet characters	`chargalois`(G)

Spaces of modular forms

Arguments of the form $[N, k, \chi]$ give the level weight and nebentypus χ; χ can be omitted: $[N, k]$ means trivial χ.

initialize $S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 0)$
initialize $S_k(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 1)$
initialize $S_k^{\mathrm{old}}(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 2)$
initialize $E_k(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 3)$
initialize $M_k(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi])$
find eigenforms	`mfsplit`(M)
statistics on self-growing caches	`getcache`$()$

We let $M =$ `mfinit`$(\ldots)$ denote a modular space.

describe the space M	`mfdescribe`(M)
recover (N, k, χ)	`mfparams`(M)
... the space identifier (0 to 4)	`mfspace`(M)
... the dimension of M over $\mathbf{C}$	`mfdim`(M)
... a $\mathbf{C}$-basis (f_i) of M	`mfbasis`(M)
... a basis (F_j) of eigenforms	`mfeigenbasis`(M)
... polynomials defining $\mathbf{Q}(\chi)(F_j)/\mathbf{Q}(\chi)$	`mffields`(M)
matrix of Hecke operator T_n on (f_i)	`mfheckemat`(M, n)
eigenvalues of w_Q	`mfatkineigenvalues`(M, Q)
basis of period poynomials for weight k	`mfperiodpolbasis`(k)
basis of the Kohnen +-space	`mfkohnenbasis`(M)
... new space and eigenforms	`mfkohneneigenbasis`(M, b)
isomorphism $S_k^+(4N, \chi) \to S_{2k-1}(N, \chi^2)$	`mfkohnenbijection`(M)

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of $S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi])$
dimension of $S_k(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 1)$
dimension of $S_k^{\mathrm{old}}(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 2)$
dimension of $M_k(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 3)$
dimension of $E_k(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 4)$
Sturm's bound for $M_k(\Gamma_0(N), \chi)$	`mfsturm`(N, k)

$\Gamma_0(N)$ cosets

list of right $\Gamma_0(N)$ cosets	`mfcosets`(N)
identify coset a matrix belongs to	`mftocoset`

Cusps

a cusp is given by a rational number or `oo`.

lists of cusps of $\Gamma_0(N)$	`mfcusps`(N)
number of cusps of $\Gamma_0(N)$	`mfnumcusps`(N)
width of cusp c of $\Gamma_0(N)$	`mfcuspwidth`(N, c)
is cusp c regular for $M_k(\Gamma_0(N), \chi)$?	`mfcuspisregular`$([N, k, \chi], c)$

Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients	`mftobasis(mf,`*vec*`)`

There are also many predefined ones:

Eisenstein series E_k on $Sl_2(\mathbf{Z})$	`mfEk`(k)
Eisenstein-Hurwitz series on $\Gamma_0(4)$	`mfEH`(k)
unary θ function (for character ψ)	`mfTheta`$(\{\psi\})$
Ramanujan's Δ	`mfDelta`$()$
$E_k(\chi)$	`mfeisenstein`(k, χ)
$E_k(\chi_1, \chi_2)$	`mfeisenstein`(k, χ_1, χ_2)
eta quotient $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$	`mffrometaquo`(a)
newform attached to ell. curve $E/\mathbf{Q}$	`mffromell`(E)
identify an L-function as a eigenform	`mffromlfun`(L)
θ function attached to $Q > 0$	`mffromqf`(Q)
trace form in $S_k^{\mathrm{new}}(\Gamma_0(N), \chi)$	`mftraceform`$([N, k, \chi])$
trace form in $S_k(\Gamma_0(N), \chi)$	`mftraceform`$([N, k, \chi], 1)$

Operations on modular forms

In this section, f, g and the $F[i]$ are modular forms

$f \times g$	`mfmul`(f, g)
f/g	`mfdiv`(f, g)
f^n	`mfpow`(f, n)
$f(q)/q^v$	`mfshift`(f, v)
$\sum_{i \le k} \lambda_i F[i]$, $L = [\lambda_1, \ldots, \lambda_k]$	`mflinear`(F, L)
$f = g$?	`mfisequal(f,g)`
expanding operator $B_d(f)$	`mfbd`(f, d)
Hecke operator $T_n f$	`mfhecke`(mf, f, n)
initialize Atkin–Lehner operator w_Q	`mfatkininit`(mf, Q)
... apply w_Q to f	`mfatkin`(w_Q, f)
twist by the quadratic char $(D/\cdot)$	`mftwist`(f, D)
derivative wrt. $q \cdot d/dq$	`mfderiv`(f)
see f over an absolute field	`mfreltoabs`(f)
Serre derivative $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$	`mfderivE2`(f)
Rankin-Cohen bracket $[f, g]_n$	`mfbracket`(f, g, n)
Shimura lift of f for discriminant D	`mfshimura`(mf, f, D)

Properties of modular forms

In this section, $f = \sum_n f_n q^n$ is a modular form in some space M with parameters N, k, χ.

describe the form f	`mfdescribe`(f)
(N, k, χ) for form f	`mfparams`(f)
the space identifier (0 to 4) for f	`mfspace`(mf, f)
$[f_0, \ldots, f_n]$	`mfcoefs`(f, n)
f_n	`mfcoef`(f, n)
is f a CM form?	`mfisCM`(f)
is f an eta quotient?	`mfisetaquo`(f)
Galois rep. attached to all $(1, \chi)$ eigenforms	`mfgaloistype`(M)
... single eigenform	`mfgaloistype`(M, F)
... as a polynomial fixed by Ker ρ_F	`mfgaloisprojrep`(M, F)
decompose f on `mfbasis`(M)	`mftobasis`(M, f)
smallest level on which f is defined	`mfconductor`(M, f)
decompose f on $\oplus S_k^{\mathrm{new}}(\Gamma_0(d))$, $d \mid N$	`mftonew`(M, f)
valuation of f at cusp c	`mfcuspval`(M, f, c)
expansion at ∞ of $f \mid_k \gamma$	`mfslashexpansion`(M, f, γ, n)
n-Taylor expansion of f at i	`mftaylor`(f, n)
all rational eigenforms matching criteria	`mfeigensearch`
... forms matching criteria	`mfsearch`

Forms embedded into C

Given a modular form f in $M_k(\Gamma_0(N), \chi)$ its field of definition $Q(f)$ has $n = [Q(f) : Q(\chi)]$ embeddings into the complex numbers. If $n = 1$, the following functions return a single answer, attached to the canonical embedding of f in $\mathbf{C}[[q]]$; else a vector of n results, corresponding to the n conjugates of f.

complex embeddings of $Q(f)$	`mfembed`(f)
... embed coefs of f	`mfembed`(f, v)
evaluate f at $\tau \in \mathcal{H}$	`mfeval`(f, τ)
L-function attached to f	`lfunmf`(mf, f)
... eigenforms of new space M	`lfunmf`(M)

Periods and symbols

The functions in this section depend on $[Q(f) : Q(\chi)]$ as above.

initialize symbol fs attached to f	`mfsymbol`(M, f)
evaluate symbol fs on path p	`mfsymboleval`(fs, p)
Petersson product of f and g	`mfpetersson`(fs, gs)
period polynomial of form f	`mfperiodpol`(M, fs)
period polynomials for eigensymbol FS	`mfmanin`(FS)

Modular Symbols

Let $G = \Gamma_0(N)$, $V_k = \mathbf{Q}[X, Y]_{k-2}$, $L_k = \mathbf{Z}[X, Y]_{k-2}$ and $\Delta = \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$. An element of Δ is a *path* between cusps of $X_0(N)$ via the identification $[b] - [a] \to$ path from a to b, coded by the pair $[a, b]$ where a, b are rationals or `oo` $= (1 : 0)$.

Let $\mathbf{M}_k(G) = \mathrm{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$; an element of $\mathbf{M}_k(G)$ is a V_k-valued *modular symbol*. There is a natural decomposition $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$ under the action of the $*$ involution, induced by complex conjugation. The `msinit` function computes either $\mathbf{M}_k$ ($\varepsilon = 0$) or its $\pm$-parts ($\varepsilon = \pm 1$) and fixes a minimal set of $\mathbf{Z}[G]$-generators (g_i) of Δ.

initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$	`msinit`$(N, k, \{\varepsilon = 0\})$
the level M	`msgetlevel`(M)
the weight k	`msgetweight`(M)
the sign ε	`msgetsign`(M)
Farey symbol attached to G	`mspolygon`(M)
... attached to $H < G$	`msfarey`(F, inH)
$H \backslash G$ and right G-action	`mscosets`$(genG, inH)$
$\mathbf{Z}[G]$-generators (g_i) and relations for Δ	`mspathgens`(M)
decompose $p = [a, b]$ on the (g_i)	`mspathlog`(M, p)

Create a symbol

Eisenstein symbol attached to cusp c	`msfromcusp`(M, c)
cuspidal symbol attached to $E/\mathbf{Q}$	`msfromell`(E)
symbol having given Hecke eigenvalues	`msfromhecke`$(M, v, \{H\})$
is s a symbol ?	`msissymbol`(M, s)

Operations on symbols

the list of all $s(g_i)$	`mseval`(M, s)
evaluate symbol s on path $p = [a, b]$	`mseval`(M, s, p)
Petersson product of s and t	`mspetersson`(M, s, t)

Operators on subspaces

An operator is given by a matrix of a fixed $\mathbf{Q}$-basis. H, if given, is a stable $\mathbf{Q}$-subspace of $\mathbf{M}_k(G)$: operator is restricted to H.

matrix of Hecke operator T_p or U_p	`mshecke`$(M, p, \{H\})$
matrix of Atkin-Lehner w_Q	`msatkinlehner`$(M, Q\{H\})$
matrix of the $*$ involution	`msstar`$(M, \{H\})$

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a $\mathbf{Q}$-basis. If H is a Hecke-stable subspace of $M_k(G)^+$ or $M_k(G)^-$, it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform $\sum_n a_n q^n$.

cuspidal subspace $S_k(G)^\varepsilon$	`mscuspidal`(M)
Eisenstein subspace $E_k(G)^\varepsilon$	`mseisenstein`(M)
new part of $S_k(G)^\varepsilon$	`msnew`(M)
split H into simple subspaces (of dim $\le d$)	`mssplit`$(M, H, \{d\})$
dimension of a subspace	`msdim`(M)
$(a_1, \dots, a_B)$ for attached newform	`msqexpansion`$(M, H, \{B\})$
$\mathbf{Z}$-structure from $H^1(G, L_k)$ on subspace A	`mslattice`(M, A)

Overconvergent symbols and p-adic L functions

Let M be a full modular symbol space given by `msinit` and p be a prime. To a classical modular symbol ϕ of level N ($v_p(N) \le 1$), which is an eigenvector for T_p with nonzero eigenvalue a_p, we can attach a p-adic L-function L_p. The function L_p is defined on continuous characters of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$; in GP we allow characters $\langle\chi\rangle^{s_1}\tau^{s_2}$, where (s_1, s_2) are integers, τ is the Teichmüller character and χ is the cyclotomic character.

The symbol ϕ can be lifted to an *overconvergent* symbol Φ, taking values in spaces of p-adic distributions (represented in GP by a list of moments modulo p^n).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that $v_p(a_p) = 0$ if *flag* $= 0$ (fastest), and that $v_p(a_p) \ge$ *flag* otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to Φ allowing to compute L_p to high accuracy.

initialize *Mp* to lift symbols	`mspadicinit`$(M, p, n, \{\mathit{flag}\})$
lift symbol ϕ	`mstooms`(Mp, ϕ)
eval overconvergent symbol Φ on path p	`msomseval`(Mp, Φ, p)
mu for p-adic L-functions	`mspadicmoments`$(Mp, S, \{D = 1\})$
$L_p^{(r)}(\chi^s)$, $s = [s_1, s_2]$	`mspadicL`$(mu, \{s = 0\}, \{r = 0\})$
$\hat{L}_p(\tau^i)(x)$	`mspadicseries`$(mu, \{i = 0\})$

Based on an earlier version by Joseph H. Silverman
November 2022 v2.38. Copyright © 2022 K. Belabas
Permission is granted to make and distribute copies of this card provided the copyright and this permission notice are preserved on all copies.

Send comments and corrections to ⟨Karim.Belabas@math.u-bordeaux.fr⟩